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A NEW BOUNDARY ELEMENT METHOD FOR BENDING OF PLATES ON ELASTIC FOUNDATIONS

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Abstract—The bending of plates on a Winkler foundation, according to Kirchhoff's theory, is solved by using an original boundary integral equation method involving the fundamental solution for plate flexure problems. An integral representation for the second member (pressure of the foundation) of the equation is given. By discretizing the integral equation, it is possible to eliminate the boundary unknowns, so that one is reduced to solving a linear system the solutions of which are deflections inside the domain. To illustrate the potentialities of this method several problems with various boundary conditions, loads and values of the modulus of the foundation are successfully solved.

1. INTRODUCTION

Boundary integral equation formulation is a powerful method for solving problems in continuum mechanics and particularly the bending of plates. Many formulations have been developed[1, 2] but the most efficient are due to the simultaneous works given in Refs [3–5]. More recently some authors have treated the very important problems of the behaviour of plates on elastic foundations by using the boundary integral equations method. The most significant works are those of Katsikadelis and Armenakas[6], Costa and Brebbia[7] and Puttonen and Varpasuo[8] which use the fundamental solution of the differential equation. This fundamental solution is a Kelvin function of the first order. Although the method gives good results it presents the disadvantage that the rigidity enters into the fundamental solution. Consequently:

(1) a change of rigidity changes the problems entirely (all the matrices should be computed once again);
(2) it is not possible to treat a problem in which the rigidity of the foundation is not a constant.

In this paper a boundary integral equation method is presented which does not have these disadvantages. The fundamental solution is the one of plate flexure problems (which does not involve the rigidity of the foundation) and the reactive force of elastic media are computed as loads per unit area.

Examples are presented and the results are compared with analytical and numerical solutions for different boundary conditions. Some examples of free boundaries are treated.

2. FORMULATION OF THE PROBLEM

Consider a plate subject to a transverse load p per unit area and let S be the interior of the plate and Γ its boundary. According to Kirchhoff's theory of thin plate bending, the transverse deflection w is governed by the following differential equation:

$$\Delta\Delta w = \frac{p}{D} \qquad \text{in } S \tag{1}$$

where Δ is the Laplacian, p the load per unit area, and D the flexural rigidity defined by $D = Eh^3/12(1-\nu^2)$, where h is the constant thickness of the plate, and E and ν Young's modulus and Poisson's ratio, respectively.

In the case of a plate resting on a Winkler-type elastic foundation, the load p is given by

$$p = -kw + \bar{p} \tag{2}$$

where k is the stiffness of the foundation and $\bar{p}$ the load applied on the plate. If the load $\bar{p}$ is a concentrated force F applied at a point P, one has $\bar{p}(Q) = F\delta(P-Q)$.

Consequently the differential equation of a plate on an elastic foundation is

$$\Delta\Delta w = -k\frac{w}{D} + \frac{\bar{p}}{D} \quad \text{in } S. \tag{3}$$

3. BOUNDARY ELEMENT METHOD FOR PLATE BENDING PROBLEMS

Considering eqn (1), the boundary element formulation is now well established[2]. The foundation of this method is the classical Rayleigh–Green identity generalized to a boundary with N corners A_i

$$\int_S (v\Delta\Delta w - w\Delta\Delta v)\,\mathrm{d}S = \frac{1}{D}\int_\Gamma \left[-vK_n(w) + \frac{\partial v}{\partial n}M_n(w) - M_n(v)\frac{\partial w}{\partial n} + K_n(v)w\right]\mathrm{d}s - \frac{1}{D}\sum_{i=1}^{N}[\![vM_{nt}(w) - M_{nt}(v)w]\!]_{A_i}. \tag{4}$$

By taking for v the singular function $v(P,Q) = (1/8\pi)r^2\log r$ one obtains the integral representation

$$\beta w(P) = \int_S \frac{p}{D}v\,\mathrm{d}S - \frac{1}{D}\int_\Gamma \left[K_n(v)w - M_n(v)\frac{\partial w}{\partial n} + \frac{\partial v}{\partial n}M^n(w) - vK_n(w)\right]\mathrm{d}s - \frac{1}{D}\sum_{i=1}^{N}[\![wM_{nt}(v) - vM_{nt}(w)]\!]_{A_i} \tag{5}$$

with $\beta = 1$ if $P\in S$ and $\beta = 1/2$ if $P\in\Gamma$ and by derivation in the n_0-direction at point P

$$\frac{1}{2}\frac{\partial w}{\partial n_0}(P) = \int_S \frac{p}{D}\frac{\partial v}{\partial n_0}\,\mathrm{d}S - \frac{1}{D}\int_\Gamma \left[\frac{\partial K_n(v)}{\partial n_0}w - \frac{\partial M_n(v)}{\partial n_0}\frac{\partial w}{\partial n} + \frac{\partial^2 v}{\partial n_0\partial n}M_n(w) - \frac{\partial v}{\partial n_0}K_n(w)\right]\mathrm{d}s - \frac{1}{D}\sum_{i=1}^{N}\left[\!\!\left[w\frac{\partial M_{nt}(v)}{\partial n_0} - \frac{\partial v}{\partial n_0}M_{nt}(w)\right]\!\!\right]_{A_i} \tag{6}$$

where $r = \|PQ\|$, P is a fixed point and Q a moving point; n is the outward normal at point Q and n_0 the outward normal at point P of Γ; $K_n(u)$ is the Kirchhoff transverse shear force associated with the deflection field u; $M_n(u)$ is the normal flexure moment associated with the deflection field u; $M_{nt}(u)$ is the torsional moment for the deflection field u and $[\![0]\!]_{A_i}$ is the jump of the function which may occur at corners A_i of curvilinear abscissa s_i defined by $[\![0]\!]_{A_i} = (0)_{s_i^+} - (0)_{s_i^-}$.

Furthermore, the quantity $M_{nt}(w)$ at point Q can be expressed in terms of $\partial w/\partial n$

$$M_{nt}(w) = -D(1-\nu)\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial w}{\partial n}. \tag{7}$$

Along the boundary the known quantities are $K_n(w)$ and $M_n(w)$ on a free edge, w and $M_n(w)$ on a simply supported edge or w and $\partial w/\partial n$ on a clamped edge. Consequently the system obtained by eqns (5) and (6) can be solved easily.

4. BOUNDARY ELEMENT METHOD FOR PLATE ON ELASTIC FOUNDATION

A first formulation of this problem consists in taking the suitable fundamental solution of eqn (3). This solution is known, it includes the Kelvin functions of the second kind, since it is

$$v(P,Q) = -\frac{1}{2\pi D}\sqrt{\left(\frac{D}{k}\right)}Kei(\rho) \tag{8}$$

with

$$\rho = r/\sqrt{(D/k)}.$$

This formulation has been used by all the authors who have treated this problem by the boundary integral equation method[6–11]. The main objection of this procedure is the difficulty in the evaluation of the integrals. It is necessary before integration to compute the Kelvin functions, for instance by their expansion in a Chebyshev series, and this for any integration point. Moreover, ρ involves the value of the stiffness foundation k, consequently all the kernels should be computed again when the value of k is modified.

The formulation proposed here uses the classical fundamental solution $v = (1/8\pi)r^2 \log r$, and replaces the pressure distribution in the foundation interface by the load applied at each node of a mesh used to discretize the plate domain.

In this way eqn (5) becomes

$$\beta w(P) = \int_S \frac{\bar{p}}{D} v \, dS - \int_S \frac{kw}{D} v \, dS - \frac{1}{D}\int_\Gamma \left[K_n(v)w - M_n(v)\frac{\partial w}{\partial n} + \frac{\partial v}{\partial n} M_n(w) - vK_n(w) \right] ds - \frac{1}{D}\sum_{i=1}^{N} [\![wM_{nt}(v) - vM_{nt}(w)]\!]_{A_i}. \tag{9}$$

To solve this new problem it is necessary to evaluate the integrals

$$\int_S k\frac{w}{D} v \, dS.$$

To do this the integral representation (5) is considered for a point P inside S. The expression of $w(P)$ is obtained at each point inside the domain, which allows the domain integrals to be performed.

5. MATRIX FORMULATIONS

5.1. *Matrix formulation of boundary integral equation*

A matrix formulation for eqns (5) and (6) can be obtained by:

(1) a discretization of the boundary into q straight elements at the middle (nodal points) of which are defined the value of deflection w, its normal derivative $\partial w/\partial n$, bending moment $M_n(w)$ and transverse shear $K_n(w)$;

(2) a discretization of the domain S in m rectangular panels at the middle (nodal points) of which are defined the value of the deflection w and the load per unit area $\bar{p}$.

For eqn (5) one obtains

$$\frac{1}{2}\{w\} = [A_\Gamma]\{K_n\}+[B_\Gamma]\{M_n\}+[C_\Gamma]\left\{\frac{\partial w}{\partial n}\right\}+[D_\Gamma]\{w\}+[E_\Gamma]\{\bar{p}-kw_S\} \tag{10}$$

and for eqn (6)

$$\frac{1}{2}\left\{\frac{\partial w}{\partial n}\right\} = [A'_\Gamma]\{K_n\}+[B'_\Gamma]\{M_n\}+[C'_\Gamma]\left\{\frac{\partial w}{\partial n}\right\}+[D'_\Gamma]\{w\}+[E'_\Gamma]\{\bar{p}-kw_S\}. \tag{11}$$

With eqns (10) and (11) the following formulation is performed:

$$[G_\Gamma]\{I\}+[J_\Gamma]\{\bar{p}\}-[J_\Gamma]\{kw_S\} = \{0\} \tag{12}$$

where $[G_\Gamma]$ is a $2q$ by $2q$ matrix, $[J_\Gamma]$ a $2q$ by m matrix, $[I]$ the vector the $2q$ components of which are the $2q$ boundary unknowns among w, $\partial w/\partial n$, $M_n(w)$ and $K_n(w)$; $\{p\}$ and $\{w\}$ are vectors of m loads and pressure in the foundation interface, respectively. Subscript Γ shows that the matrices are obtained in the case of points P belonging to the boundary.

5.2. *Matrix formulation of deflection inside S*

In the same way as eqn (12) the identity (5) for P inside S can be written following a matrix formulation:

$$\{w_S\} = [G_S]\{I\}+[J_S]\{\bar{p}\}-[J_S]\{kw_S\} \tag{13}$$

where $[G_S]$ is an m by q matrix and $[J_S]$ an m by m matrix.

The plate bending on an elastic foundation problem consists in solving simultaneously eqns (12) and (13), since one has $(2q+m)$ equations with $(2q+m)$ unknowns. Nevertheless it is more useful to modify the formulation by the elimination of boundary unknowns $\{I\}$.

6. ELIMINATION OF BOUNDARY UNKNOWNS

It is possible to solve the system of eqns (12) and (13) which comprises $(2q+m)$ unknowns, but this leads to numerous equations and their treatment is expensive. In this case a more convenient method consists of eliminating the unknowns on the boundary, so as to obtain a smaller system of m equations for the m unknowns inside the domain.

In this way one can invert the matrix $[G_\Gamma]$ to obtain from eqn (12)

$$\{I\} = -[G_\Gamma^{-1}][J_\Gamma]\{\bar{p}\}-[G_\Gamma^{-1}][J_\Gamma]\{kw_S\} \tag{14}$$

where $[G_\Gamma^{-1}]$ is the inverse matrix of $[G_\Gamma]$.

By substituting eqn (14) in eqn (13) one obtains

$$\begin{aligned}\{w_S\} = &-[G_S][G_\Gamma^{-1}][J_\Gamma]\{\bar{p}\}+[J_S]\{\bar{p}\}\\ &-[G_S][G_\Gamma^{-1}][J_\Gamma]\{kw_S\}+[J_S]\{kw_S\}\end{aligned} \tag{15}$$

which can be written in the following linear system of m equations as:

$$(k[\mathbf{K}]+[\mathbf{1}])\{w_S\} = -[\mathbf{K}]\{\bar{p}\} \tag{16}$$

where

$$[\mathbf{K}] = [G_S][G_\Gamma^{-1}][J_\Gamma]-[J_S] \tag{17}$$

and [1] is the identity matrix. When system (16) is solved, the boundary unknowns can be easily obtained by computing eqn (14).

7. NUMERICAL RESOLUTION

The boundary Γ is approximated by a succession of straight segments q_i of centre Q_i. On each segment one has integrals such as

$$\int_{q_i} f(P_0, Q) g(Q)\, \mathrm{d}s_Q$$

where $g(Q)$ stands for one of the unknowns w, $\partial w/\partial n$, M_n or K_n. This unknown is supposed to be constant over each segment, its value being that at the centre Q_i of the segment and the integration of

$$\int_{q_i} f(P_0, Q)\, \mathrm{d}s_Q$$

is carried out by a Gauss–Legendre integration method with ten points.

The domain S is divided into m finite panels. The values of the deflection w and of the load per unit area $\bar{p}$ are defined at the centre point of each panel. Thus taking w and $\bar{p}$ constant over each panel the surface integrations of kernels are also performed by a Gauss–Legendre method with 6×6 integration points.

8. NUMERICAL RESULTS

As applications of the previous formulation rectangular plates on elastic foundation were studied.

For each problem Poisson's ratio is taken to be 0.3 and all the results for dimensionless variables x/a and y/b are given where $2a$ and $2b$ are the side lengths and the origin is located at the centre of the plate.

In all the cases the boundary has been divided into 48 straight segments and the domain has been divided into 49 (7×7) or 77 (11×7) rectangular panels, respectively, when $b/a = 1$ or 1.6.

Boundary conditions are clamped, simply supported or free edges. Results presented are deflections inside the domain, M_n and K_n along the edges.

These results are obtained for different values of k which varies between 0 (does not rest on elastic foundation) and $2500D/a^4$.

8.1. *Clamped plate*

This example has been treated by Costa and Brebbia[7] and with a Galerkin variational method by Ng[12] for a uniformly loaded square plate.

Results for the variation of centre deflection, and at the middle of the edge for bending moment $M_n(w)$ and transverse shear force $K_n(w)$ with various modulus k are shown in Figs 1–3, respectively. One can see that for $k = 200D/a^4$ the results are in good agreement with those given in Refs [7, 12] since for centre deflection one has a difference of 5%, and 7% for the bending moment.

In Fig. 4 the present results are compared with those of Ng[12] for the variation of centre deflection with b/a varying between 0 and 2. It can be seen that these results are in good agreement.

8.2. *Simply supported plate*

For this boundary condition the present results are compared with those of Katsikadelis and Armenakas[6] for a $2a \times 2b$ rectangular plate with $b/a = 1.6$. For the modulus coefficient $k = 625D/a^4$ in Table 1 are given the value of the deflections at the plate centre and the

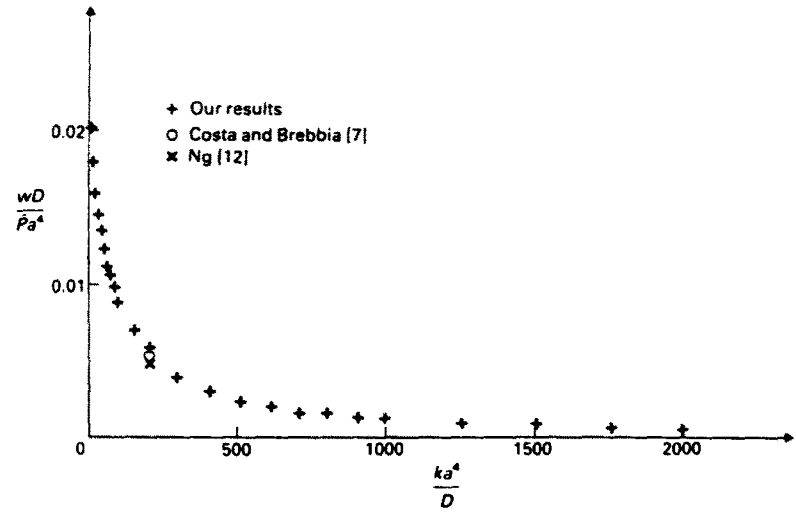

Fig. 1. Clamped plate ($2a \times 2a$): variation of centre deflection with k for a load per unit area $\bar{p}$.

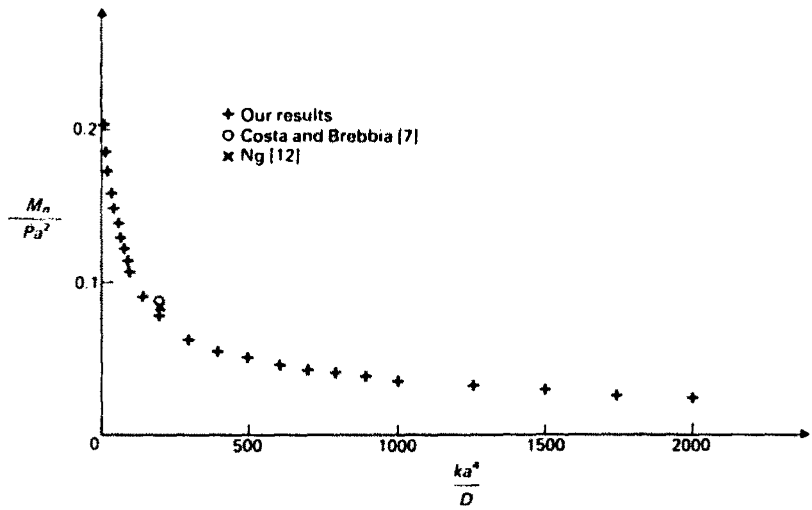

Fig. 2. Clamped plate ($2a \times 2a$): variation of bending moment $M_n(w)$ at a middle side with k for a load per unit area $\bar{p}$.

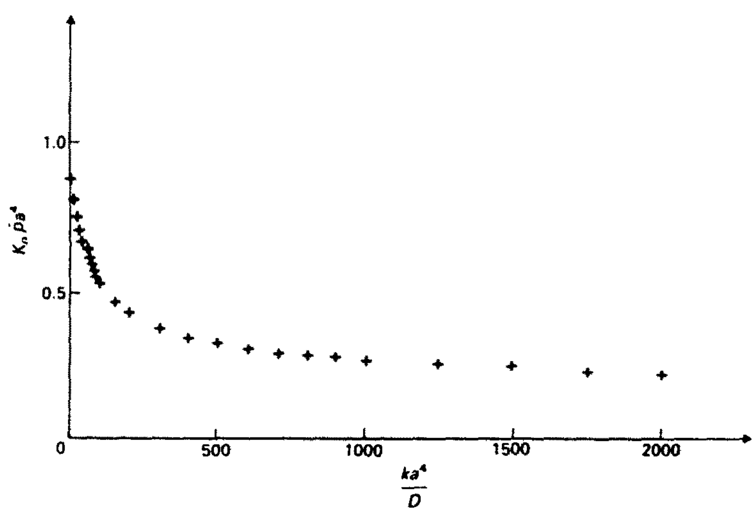

Fig. 3. Clamped plate ($2a \times 2a$): variation of transverse shear force $K_n(w)$ at a middle side with k for a load per unit area $\bar{p}$.

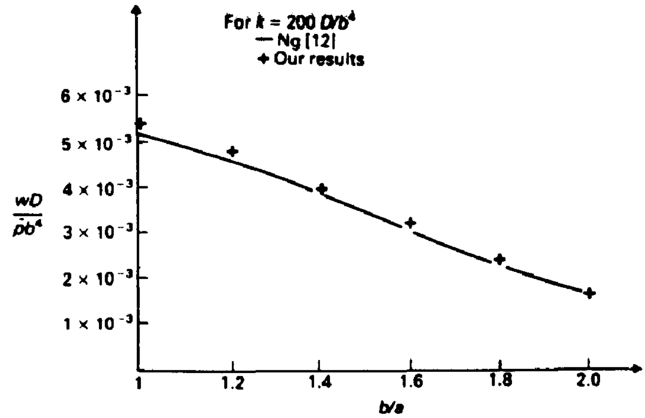

Fig. 4. Variation of maximum centre deflection with elastic support and aspect ratio for rectangular plates.

Table 1. Simply supported plate ($2a \times 2b$, $b/a = 1.6$): comparison of deflection centre values with Ref. [6] following the application point (X_F, Y_F) of the concentrated load

Y_F/b		X_F/a 0	0.2	0.4	0.6	0.8
0	Present work	0.503×10^{-2}	0.324×10^{-2}	0.131×10^{-2}	0.328×10^{-3}	0.168×10^{-4}
	[6]	0.500×10^{-2}	0.315×10^{-2}	0.129×10^{-2}	0.343×10^{-3}	0.320×10^{-4}
	Percentage	7.5	3	1.5	4	—
0.2	Present work	0.194×10^{-2}	0.149×10^{-2}	0.649×10^{-2}	0.135×10^{-3}	-0.251×10^{-4}
	[6]	0.192×10^{-2}	0.145×10^{-2}	0.649×10^{-2}	0.150×10^{-3}	-0.105×10^{-4}
	Percentage	1	3	0.5	10	—
0.4	Present work	0.200×10^{-3}	0.139×10^{-3}	0.832×10^{-5}	-0.695×10^{-4}	-0.606×10^{-4}
	[6]	0.217×10^{-1}	0.152×10^{-1}	0.266×10^{-4}	-0.499×10^{-4}	-0.484×10^{-4}
	Percentage	8	8	—	—	—
0.6	Present work	-0.855×10^{-4}	-0.858×10^{-4}	-0.817×10^{-4}	-0.653×10^{-4}	-0.426×10^{-4}
	[6]	-0.732×10^{-4}	-0.742×10^{-4}	-0.719×10^{-4}	-0.582×10^{-4}	-0.325×10^{-4}
	Percentage	16	16	14	12	—
0.8	Present work	-0.393×10^{-4}	-0.376×10^{-4}	-0.314×10^{-4}	-0.230×10^{-4}	-0.173×10^{-4}
	[6]	-0.361×10^{-4}	-0.340×10^{-4}	-0.281×10^{-4}	-0.195×10^{-4}	-0.982×10^{-5}
	Percentage	9	10	12	18	—

difference with Katsikadelis and Armenakas' results, obtained for a concentrated load successively applied in points of coordinates (αa, βb) ($\alpha = 0, 0.2, 0.4, 0.6, 0.8$, and $\beta = 0, 0.2, 0.4, 0.6, 0.8$).

One can see very good accuracy for values of α and $\beta < 0.6$. When α or $\beta = 0.6$, results show a difference up to 20% but it is obvious that values of deflections are very small, and the disparity becomes greater than for $\alpha = \beta = 0$. In fact for instance, deflection for $\alpha = 0.2$, $\beta = 0.6$ is 2% of the deflection for $\alpha = \beta = 0$.

Finally in Fig. 5 the deflection along a symmetry axis is given for $k = 50D/a^4$, $200D/a^4$, $500D/a^4$ and $1500D/a^4$.

8.3. *Cantilever plate*

In this example the problem of a free boundary is solved. Consider a square plate ($2a \times 2b$, $b/a = 1$) loaded by a load per unit area. In Fig. 6 is given the value of deflection at the point $x/a = 1$, and $y/b = 0$ (end of the symmetry axis). In Fig. 7 deflections along the symmetry axis are shown for four values of k ($k = 50D/a^4$, $200D/a^4$, $500D/a^4$ and $1500D/a^4$).

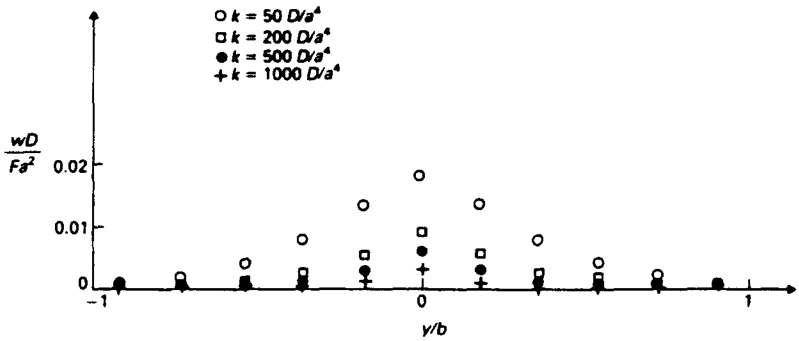

Fig. 5. Simply supported plate ($2a \times 2b$, $b/a = 1.6$): values of deflection on the symmetry axis for a concentrated load at the centre.

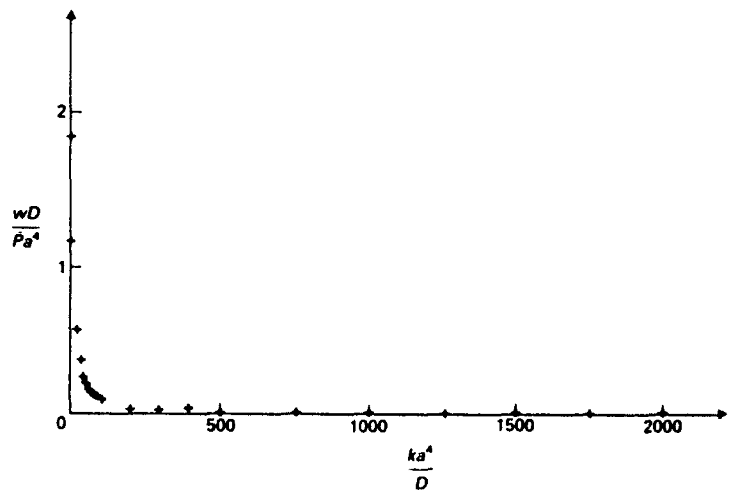

Fig. 6. Cantilever plate ($2a \times 2a$): deflection at the point ($x/a = 1$, $y/a = 0$) with k for a load per unit area.

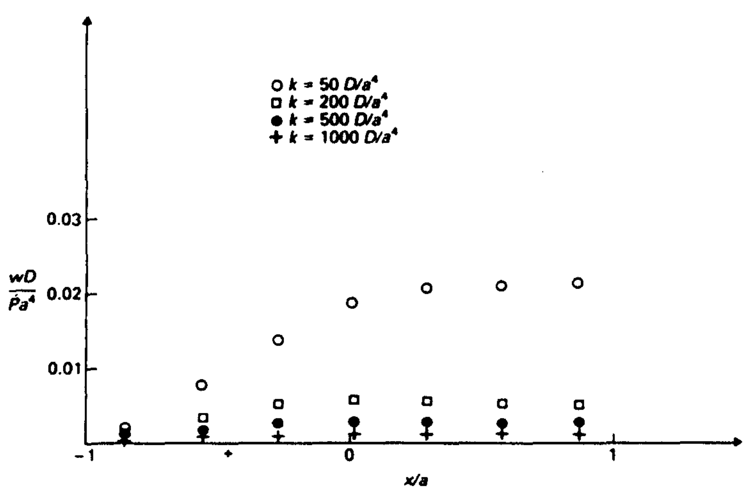

Fig. 7. Cantilever plate ($2a \times 2a$): values of deflection on the symmetry axis for a load per unit area.

9. CONCLUSION

From this study it can be concluded that the previous formulation leads to accurate results. The examples treated are not exhaustive since any problems with mixed boundary conditions can be studied whatever the domain's shape. This method gives a systematic procedure to solve the plate on Winkler foundations, and presents all the advantages of the boundary integral equation method. Moreover, as opposed with earlier works, it can be easily extended to foundations with a non-constant rigidity, or to unilateral elastic foundation ($w \geqslant 0$), by replacing k in eqns (12) and (13) by a diagonal matrix and using if necessary an iterative process.

Finally if one wants to modify the rigidity of the foundation, it is not necessary to compute all the matrices in eqns (14) and (15), but only to calculate a new matrix ($k[\mathbf{K}]+[\mathbf{1}]$) and to solve the linear system, since k is not included in the fundamental solution contrary to other integral formulations.

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